Bayesian inference to study low-level radioactivity in the environment: Application to the detection of xenon isotopes of interest for the CTBTO

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Introduction

• The **decision** that a given detection level corresponds to the effective presence of a radionuclide, or to its absence, is **widely made using a classic hypothesis test (Currie)**.

• **Shortcomings of the classic framework** / **promising perspectives of Bayesian statistics** (part of recent standards [ISO 11929-7, 2005])

⇒ we propose a **novel Bayesian approach** providing estimates of:

– the **probability of zero radioactivity**, together with **physically meaningful point and interval estimates**,

– the **prior density of the radioactivity**, obtained by **fitting previously recorded radioactivity data**.

The classic framework [Currie, 1968]

True net radioactivity level μ?
Observed net count : $x = x_g - x_b$
with blank count $x_b \rightarrow$ Poisson(μ_b) and gross count $x_g \rightarrow$ Poisson($\mu + \mu_b$)
Independence $\Rightarrow$ $E(x) = \mu$, $\mathrm{var}(x) = \sigma^2 = \mu + 2\,\mu_b$
Large enough counting time $\Rightarrow \approx$ Gaussian CDF: $F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$

Test of H_0: $\mu = 0$ against H_1: $\mu > 0$ with a type I error (false alarm) risk α
Currie's critical level L_C:

$$L_C = \Phi^{-1}(1-\alpha)\sqrt{2x_b}$$

- if **$x > l_c$**, H_0 is rejected with error risk α. 1 – α confidence interval:

$$\mu \in x \pm \Phi^{-1}(1-\alpha/2)\sqrt{x_b + x_g}$$

$\Rightarrow$ **the confidence interval may include negative values**

- if **$x \leq l_c$**, H_0 is accepted. Currie's detection limit L_D:

$$L_D = 2 \times L_C + \left(\Phi^{-1}(1-\alpha)\right)^2$$

$\Rightarrow$ **unknown probability of missing a real event** (type II error)

No use of <u>past</u> observed values of x and x_b.

The Bayesian framework (1)

True radioactivity $\mu \equiv$ random variable

$\Rightarrow$ "H_0 is true" ($\mu = 0$) and "H_1 is true" ($\mu > 0$) form a complete set of events, with *a priori* and *a posteriori* (given the observed net count x) probabilities:

$$\text{posterior probability} \longrightarrow \boxed{P(H_i\,|\,x)} = \frac{f(x\,|\,H_i)P(H_i)}{\sum_{k=0,1} f(x\,|\,H_k)\boxed{P(H_k)}} \longleftarrow \text{prior probabilities}$$

Medical diagnosis

– $H_0 \equiv$ "healthy", $H_1 \equiv$ "sick", $x \equiv$ result of a medical test ("+" or "–")

– the ***a priori* probability $P(H_0)$** and the ***conditional* probabilities $f(x|H_k) \equiv P(x|H_k)$** are estimated with the frequencies observed on a large representative sample of patients $\Rightarrow$ **no theoretical problem**

Radioactivity detection

Real-valued x depending on H_k, i.e. on $\mu \Rightarrow$ **conditional densities $f(x|H_k)$**

$$f(x\,|\,H_k) = \int_{\mu \in \Theta_k} f(x\,|\,\mu)\pi(\mu\,|\,H_k)d\mu$$

– $\Theta_0 = \{0\}$ (under H_0) and $\Theta_1 =]\,0\,;\,+\infty[$ (under H_1)

– $f(x|\mu)$: the Gaussian density (= $\varphi(\,(x-\mu)/\sigma\,)$)

– **$\pi(\mu|H_k)$: *a priori* density of the activity μ under H_k**

The Bayesian framework (2)

Denominator of the a *posteriori* probabilities = **prior density of the true radioactivity μ**:

$$\pi(\mu)=\sum_{k=0,1}\pi(\mu\,|\,H_k)P(H_k)$$

If $\pi(\mu)$ can be estimated:

- **marginal density of the net count x**: $f(x)=\int f(x\,|\,\mu)\pi(\mu)d\mu$
- **posterior probability of H_0 (no radioactivity)**: $\boxed{P(H_0\,|\,x)=\frac{f(x\,|\,\mu=0)P(H_0)}{f(x)}}$
- posterior density μ: $f(\mu\,|\,x)=\frac{f(x\,|\,\mu)\pi(\mu)}{f(x)}$
- **point estimate of the true radioactivity**:

$$\boxed{\mu^{*}=E(\mu\,|\,x)=\int\mu f(\mu\,|\,x)d\mu}$$

- **$1-\gamma$ credibility interval** $[\mu^{-}\,;\,\mu^{+}]$:

$$\boxed{\frac{\gamma}{2}=\int_{\mu^{+}}^{+\infty}f(\mu\,|\,x)d\mu=\int_{-\infty}^{\mu^{-}}f(\mu\,|\,x)d\mu}$$

positive values only

Bayesian priors (1)

The prior density of the true radioactivity μ should be given by:

$$\pi(\mu) = P(H_0)\,\delta_0(\mu) + P(H_1)\,\pi(\mu\,|\,H_1)$$

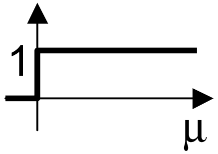
probability of no radioactivity

Dirac peak at zero

radioactivity density under H_1

Improper prior approach [Zähringer & Kirchner, 2008]:

$$\pi(\mu) = \mathbb{I}_{[0;+\infty[}(\mu)$$



– no Dirac peak at zero ⇒ **$P(H_0|x) = 0$ whatever the observed x**

– not integrable ⇒ **the marginal density f(x) cannot be estimated**

Implicit prior approach [Vivier et al., 2009]:

– $P(H_0|x) = 1 - \Phi(x/\sigma)$ ⇒ **$P(H_0|x=0) = 0.5$ whatever σ (depends on μ_b)**

– no explicit prior ⇒ **the marginal density f(x) cannot be estimated**

Bayesian priors (2)

Proposed prior (proper):

Dirac peak at zero

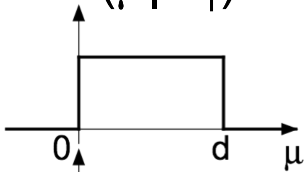
$$\pi(\mu) = P(H_0)\,\delta_0(\mu) + P(H_1)\,\pi(\mu\,|\,H_1)$$

probability of no radioactivity

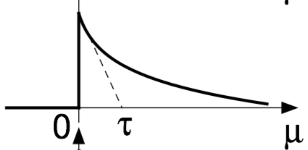
radioactivity density under H_1

Possible functional forms for $\pi(\mu|H_1)$:

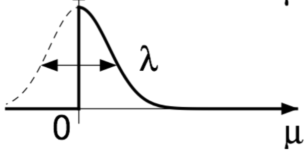
- uniform (d > 0):



- exponential (τ > 0):



- half-Gaussian (λ > 0):

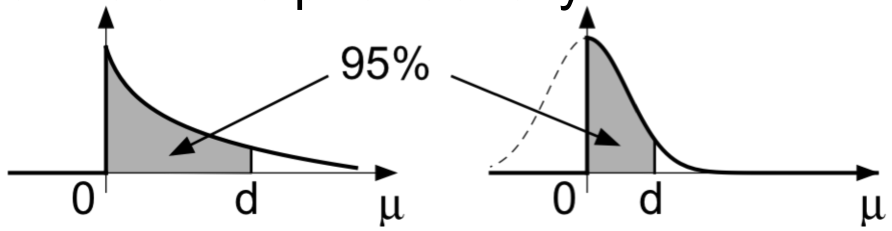


We calculated all the Bayesian estimates for these three priors.

Estimate $P(H_0)$ and $p(\mu|H_1)$, i.e. the parameter d, τ or λ by fitting the marginal density $f(x) = \int f(x|\mu)\,\pi(\mu)\,d\mu$ to past records of observed x.

Empirical fit of the prior parameters

- d, τ, and λ are merged into **a single parameter, d**, by taking τ and λ so that d is the 95th centile of the prior density:
- **$P(H_0) \equiv p_0$**



Expression of the marginal density of x (uniform prior):

$$f(x)=\frac{p_0}{\sigma}\varphi\left(\frac{x}{\sigma}\right)+\frac{1-p_0}{d}\left(\Phi\left(\frac{x}{\sigma}\right)-\Phi\left(\frac{x-d}{\sigma}\right)\right)$$

In fact, **$\sigma^2 = \mu + 2\mu_b$**. Numerical simulations show that all the results hold when σ^2 varies by simply replacing σ^2 by $x + 2\mu_b$.

Fit f(x) to past records of x with maximum likelihood, two options:

– as a **function of p_0, d and μ_b**,

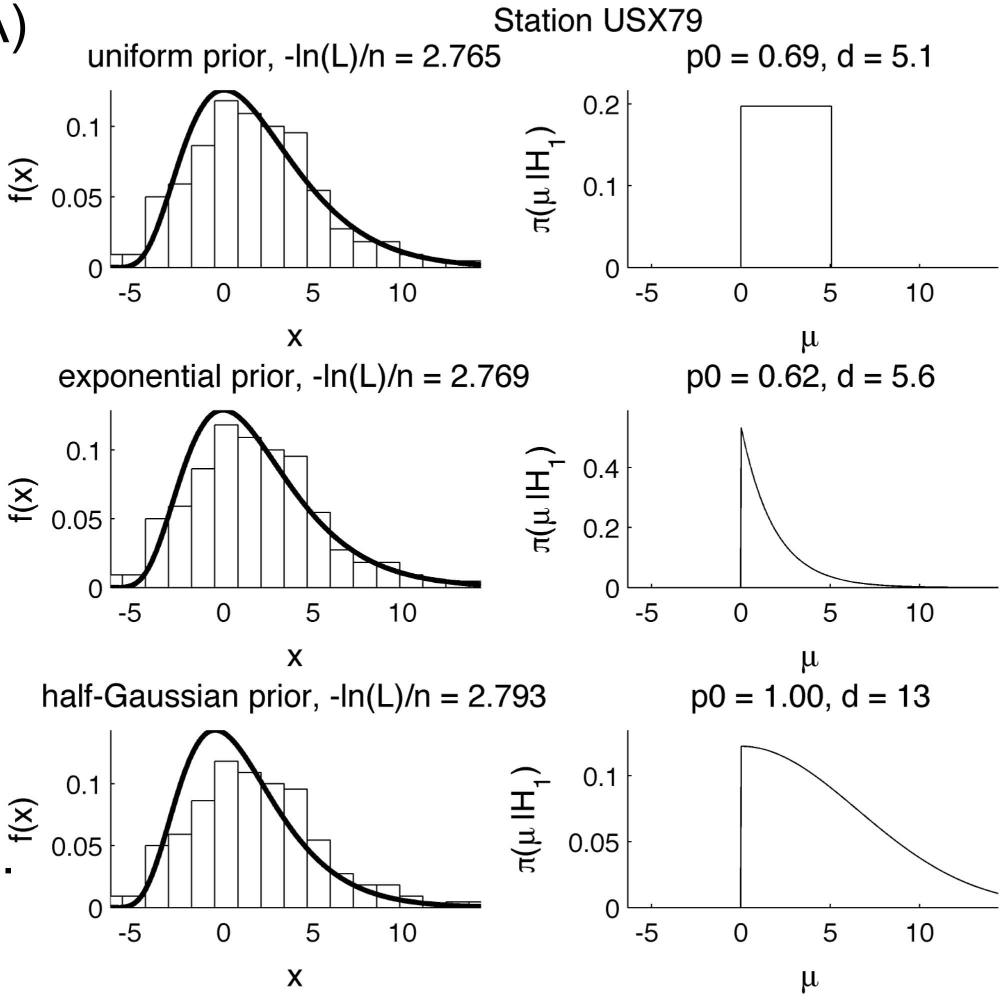
– first estimate μ_b using measured values of x_b ⇒ **check whether $\mu_b \approx$ cte** then fit f(x) as a **function of p_0 and d only (chosen option)**.

Experimental results: empirical fit of the priors

Six month of 2009 daily measurements for a dozen of CTBTO stations:
– Hawaï station USX79 (SAUNA)
– Xe131m data

- Poisson distribution of blank x_b with $\mu_b \approx 4$

- ML fit of the marginal density f(x) of the observed net counts ⇒ **good results with the uniform and exponential priors**: there is an *a priori* probability of $p_0 \approx 2/3$ of a small net radioactivity ($d \approx 5\text{-}6$).

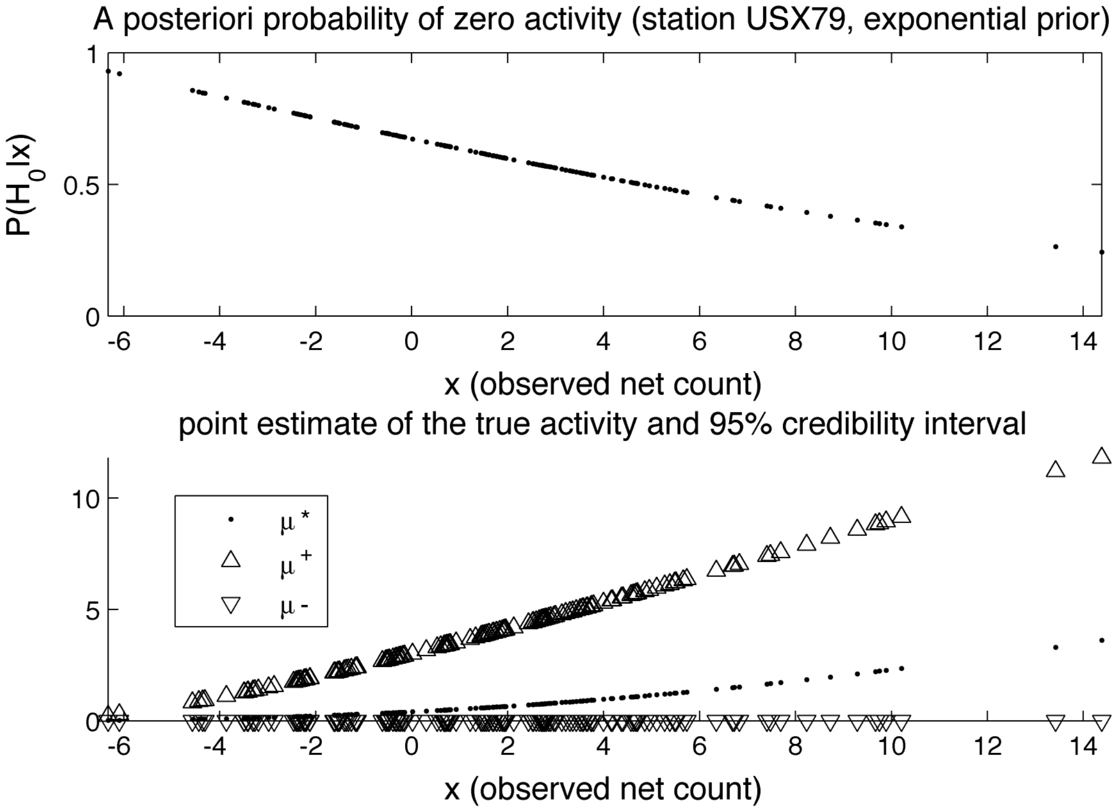


Experimental results: Bayesian estimates

For each observed net count x:

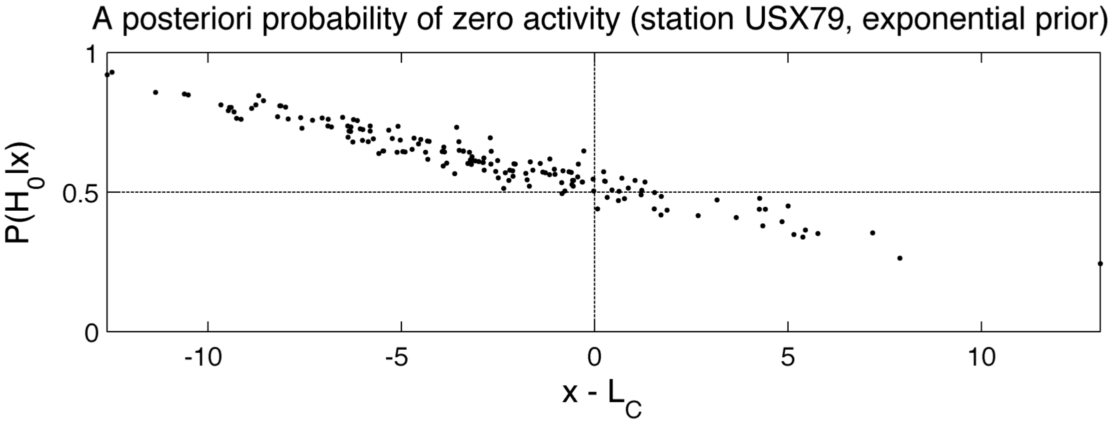
– **estimate of the posterior probability of zero radioactivity** $P(H_0|x)$

– **point estimate** μ^*, **credibility interval** $[\mu^+ ; \mu^-]$ **for the true activity** μ



Experimental results: comparison to the classic framework

The posterior probability of zero activity $P(H_0|x)$ is now shown as a function of $x – L_C$ (L_C is Currie's critical level): $x – L_C > 0$: reject H_0, else accept H_0



⇒ $x – L_C > 0$ (reject H_0) coincides with $P(H_0|x) < 0.5$

In the Bayesian framework, H_0 should be rejected when:

$$P(H_0|x) < C_{II}/(C_I+C_{II})$$

where C_I and C_{II} are the costs associated to type I (**false alarm**) and type II (**missing a real event**) errors respectively ⇒ **quantifying these costs would define the decision threshold for CTBTO.**

Conclusion

Realizations

Novel approach based on rigorous Bayesian principles (proper priors):

– probability estimate of a truly radioactive sample

– physically meaningful estimates of its radioactivity

– *a priori* knowledge is taken into account via the fit of the prior to data observed in the past

Outlooks

Extend the method to:

- time-varying blank level
- analysis of several isotopes jointly (Xe131m, Xe133m, Xe133, Xe135)
- analysis of other fission and activation products (aerosols) of interest for CTBTO

Supplementary slide 1: references

- **L. A. Currie** (1968)
Limits for qualitative detection and quantitative determination; application to radiochemistry
Anal. Chem. **335**, 586-593.

- **ISO 11929-7** (2005)
Determination of the Detection Limit and Decision Threshold for Ionizing Radiation Measurements, Part 7: Fundamentals and General Applications.

- **M. Zähringer & G. Kirchner** (2008)
Nuclide ratios and source identification from high-resolution gamma-ray spectra with Bayesian decision methods.
Nuclear Instruments and Methods in Physics Research A **594**, 400-406.

- **A. Vivier, Gilbert Le Petit, B. Pigeon & X. Blanchard** (2009)
Probabilistic assessment for a sample to be radioactive or not: application to radioxenon analysis
Journal of Radioanalytical and Nuclear Chemistry **282**, 743–748.

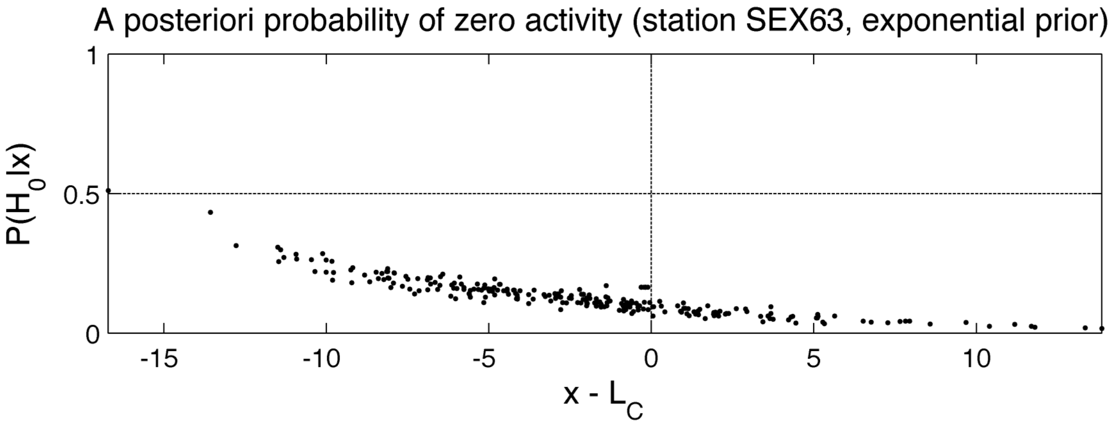
- **I. Rivals, C. Fabbri, G. Euvrard & X. Blanchard**
A Bayesian method with empirically fitted priors for the evaluation of environmental radioactivity: application to low-level radioxenon measurements
Submitted to *Nuclear Instruments and Methods in Physics Research A*.

Supplementary slide 2: results for Stockholm station SEX63

Constant low blank level **$\mu_b \approx 6$**

Good fit with the exponential prior: **p_0 = 0.13** (low *a priori* probability of no radioactivity), **d = 7.1** (small activity when present)

⇒ consistent with the station location in northern Europe (vicinity of nuclear power plants)



⇒ with a threshold of 0.5 for $P(H_0|x)$, **the proposed Bayesian approach detects more radioactive events than the classic approach**.

Supplementary slide 3: validity of the Gaussian approximation even for very small counts

Let X → Poisson(a) and Y → Poisson(b), and Z = X – Y. Then:

$$P(Z=z)=e^{-(a+b)}\left(\frac{a}{b}\right)^{z/2} I_{|z|}\left(2\sqrt{ab}\right) \quad \text{with} \quad I_z(u)=(u/2)^k\sum_{j=0}^{+\infty}\frac{(u/2)^{2j}}{j!(j+z)!}$$

Application to:

- gross count: $X_g \rightarrow \text{Poisson}(\mu_g = 8)$
- blank count: $X_b \rightarrow \text{Poisson}(\mu_b = 5)$
- net count: $X = X_g - X_b$ ($\mu = 3$)